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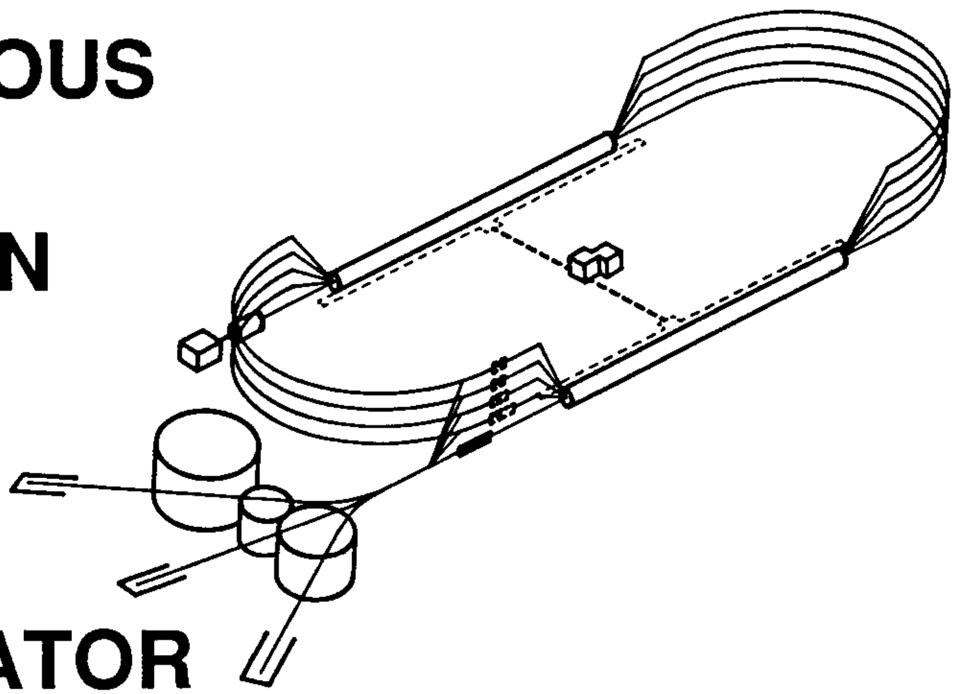

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ABSTRACT

Since space charge waves on a particle beam exhibit both dispersive and nonlinear character, soliton-like behavior is possible. Some theoretical aspects of dispersive, nonlinear wave propagation in high brightness beams are discussed. Numerical examples for realizable beams are presented, and issues for future studies are noted.

INTRODUCTION

Space charge forces can produce longitudinal density waves in low momentum spread, charged particle beams.[1] For a uniform beam of radius a transported in a perfectly conducting beampipe of radius b, the propagation is nondispersive in the linear, long wavelength approximation. The wave velocity v_p is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{e^2 \lambda_0 g}{4\pi\epsilon_0 m}} \tag{1}$$

where ω is the mode frequency for wave number k, e is the electron charge, λ_0 is the unperturbed linear particle density, $g = 1 + 2\ \log b/a$, ϵ_0 is the permittivity of free space, and m is the mass of the beam particles. However, for large density perturbations nonlinearity cannot be ignored, and for short wavelengths (small compared to the beampipe dimension) the propagation is dispersive with the wave velocity dependent on wavelength. For many physical systems[2] this combination of nonlinearity and dispersion leads to solitary waves and solitons. In fact, this is the case for the illustrative particle beam configuration discussed in this paper.

SOLITARY WAVES AND SOLITONS

Nonlinearity in wave propagation typically leads to steepening phenomena. For example, consider the simple[3] wave equation

$$u_t + (1+u)u_x = 0 \tag{2}$$

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which has the implicit solution

$$u(x,t) = f(x - (1+u)t) \tag{3}$$

where f is an arbitrary differentiable function. Note the velocity, $(1+u)$, depends on the amplitude, and, in particular, higher amplitudes propagate faster. If f describes a localized distribution, the peak value will tend to overtake lower values, and steepening and breaking of the pulse will result. On the other hand, if the velocity depends strongly on wavelength (dispersion), a localized distribution spreads as it propagates. A solitary wave results when the nonlinear steepening is canceled by the dispersive spreading, yielding a localized disturbance which propagates without distortion. Since solitary waves of different heights will generally travel with different velocities, collisions can occur. The term soliton describes solitary waves which maintain their identity and shape after collision.

SPACE CHARGE FORCES

For a beam in a beampipe, the longitudinal force F generated by longitudinal density variations is described by

$$F = \frac{-ge^2}{4\pi\epsilon_0}\frac{\partial\lambda}{\partial z} \tag{4}$$

in the long wavelength limit for density λ. In k-space, the spatial Fourier transform $\tilde{F} \propto ik\tilde{\lambda}$. More generally, the Green's function for a cylindrically symmetric distribution in a cylindrical symmetric pipe is

$$G(\rho, z; \rho', z') = \frac{1}{4\pi\epsilon_0}\frac{2}{\pi b^2}\int_{-\infty}^{\infty} dk \sum_{n=1}^{\infty}(-ik)e^{ik(z-z')}\frac{J_0(\frac{x_n\rho}{b})J_0(\frac{x_n\rho'}{b})}{((\frac{x_n}{b})^2 + k^2)J_1^2(x_n)} \tag{5}$$

where x_n is the n^{th} zero of the Bessel function J_0. Note that for small $k << (x_n/b)$, ik behavior dominates.

Consider a distribution of the form $J_0(x_1\rho/b)e^{ikz}$. In a linearized fluid model, this function describes a perturbation eigenmode of a uniform beam filling the beampipe. The underlying force law is modified from

$$ik \longrightarrow \frac{ik}{1+\alpha k^2} \tag{6}$$

where $\alpha = b^2/x_1^2$. The phase velocity

$$v_{\text{phase}} = \frac{v_p}{\sqrt{1+\alpha k^2}} \tag{7}$$

where the g implicit in v_p is now a geometric factor of order unity, and the propagation has become dispersive. On expanding the denominator of the right side of relation (6) for small α, we note that a third derivative term $(-ik^3)$ is added to the first derivative term (ik). This is suggestive of the structure of the Kortweg-DeVries (KdV) equation, which exhibits soliton behavior.

1-D NONLINEAR FLUID MODEL

As a first step in understanding the interplay of nonlinearity and dispersion for space charge dominated beams, we analyze a 1-D nonlinear cold fluid model of a uniform beam with the force law given in relation (6). Admittedly, some possibly important transverse effects may be lost. With $v_p = 1$, the fluid equations are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0 \tag{8}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{\partial \Phi}{\partial x} \tag{9}$$

$$n(x,t) = n_0 + n_1(x,t) \tag{10}$$

$$\tilde{\Phi}(k) = \frac{\tilde{n}_1}{1+\alpha k^2}\ . \tag{11}$$

At this point we can parallel Davidson's discussion of ion-acoustic solitary waves,[4] and look for solutions of the form $n_1(qx - \omega t)$, $v(qx - \omega t)$, etc. which roll-off at $\pm\infty$. Equations (8)-(11) imply that

$$n = \frac{n_0}{1 - \frac{q}{\omega}v} \tag{12}$$

$$\left(\frac{\omega}{q}\right)^2 = \left(\frac{\omega}{q} - v\right)^2 - 2\Phi \tag{13}$$

and for localized Φ

$$\frac{\alpha q^2 \Phi'^2}{2} - \frac{\Phi^2}{2} - n_0\left(\frac{\omega}{q}\right)^2\sqrt{1 - 2\left(\frac{q}{\omega}\right)^2\Phi} = 0 \tag{14}$$

where $'$ denotes differentiation. The resulting first order equation (14) is easily solved numerically for Φ, n, and v to yield the pulse shape of the self-consistent solitary waves as a function of the parameter ω/q. The peak value of Φ is given by

$$\Phi_{\mathrm{peak}} = 2\left(\frac{\omega}{q} - 1\right) \tag{15}$$

and the peak density is given by

$$n_{\mathrm{peak}} = \frac{n_0}{1 - 2\left(\frac{q}{\omega}\right)^2\Phi}\ . \tag{16}$$

When $\omega/q = 2$, $\Phi = 2$, and the density n becomes singular, indicating breaking.

A multiple time scale analysis of these fluid equations with (ω/q) as the small expansion parameter yields the KdV equation as the lowest approximation. The KdV soliton, however, does not exhibit breaking. This difference for large (ω/q) is traceable to the weaker high frequency dispersion associated with the

$$\frac{k}{1+\alpha k^2} \tag{17}$$

behavior of the space charge force versus the

$$k - \alpha k^3 \tag{18}$$

behavior implicit in the KdV equation.

CONCLUSIONS

A simple, 1-D model of longitudinal space charge waves exhibits solitary waves together with breaking at large amplitudes. Clearly, this analysis represents only a first step in understanding, and many questions remain open. Of most importance are the complications introduced by the transverse distribution and betatron oscillations. Although $J_0(x_1\rho/b)e^{ikz}$ provides a self-consistent mode for the linearized equations, this transverse distribution is not self-consistent for the nonlinear system. The full Green's function, with the infinite sum exhibited in equation (5), needs to be addressed. Also, the assumption of transport of a high current beam of the same dimension as the beampipe simplified the mathematics (collapsing the infinite sum), but it is not practical experimentally. Wall resistance and the associated slow growing instability would complete the picture.

Whether these solitary waves are indeed solitons is not clear, even in the 1-D model presented. Whitham[4] has studied a similar force law in a model of water waves and found preservation of wave shape after the collision of two such localized pulses. He also found some interesting phenomena associated with breaking. Both one and two dimensional simulations would be valuable in investigating these issues more thoroughly.

The scaling of possible experiments is set by the parameter v_p given in equation (1). For example, breaking occurs when $\omega/q = 2$ in units of v_p, and the solitary wave velocity lies between v_p and $2v_p$. Low energy ($\beta = 0.3$) electron beams[5] found in high-space-charge transport experiments can take values of v_p approaching 10^7 m/s. Since dispersive effects are expected for pulse lengths of the order of the beampipe radius, typically centimeters, it appears that several meters of transport may be sufficient to observe some of the phenomena discussed. Ion storage rings may also offer some possibilities, although the microwave instability could be a problem.

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